

OPTIMAL EXTENSIONS FOR p -TH POWER FACTORABLE OPERATORS

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ABSTRACT. Let $X(\mu)$ be a function space related to a measure space (Ω, Σ, μ) with $\chi_\Omega \in X(\mu)$ and let $T: X(\mu) \rightarrow E$ be a Banach space valued operator. It is known that if T is p -th power factorable then the largest function space to which T can be extended preserving p -th power factorability is given by the space $L^p(m_T)$ of p -integrable functions with respect to m_T , where $m_T: \Sigma \rightarrow E$ is the vector measure associated to T via $m_T(A) = T(\chi_A)$. In this paper we extend this result by removing the restriction $\chi_\Omega \in X(\mu)$. In this general case, by considering m_T defined on a certain δ -ring, we show that the optimal domain for T is the space $L^p(m_T) \cap L^1(m_T)$. We apply the obtained results to the particular case when T is a map between sequence spaces defined by an infinite matrix.

1. INTRODUCTION

Although the concept of p -th power factorable operator have previously been used as a tool in operator theory, it was introduced explicitly in [19, § 5]. Given a measure space (Ω, Σ, μ) and a Banach function space $X(\mu)$ of $(\mu$ -a.e. classes of) Σ -measurable functions such that $\chi_\Omega \in X(\mu)$, for $1 \leq p < \infty$, a Banach space valued operator $T: X(\mu) \rightarrow E$ is p -th power factorable if there is a continuous extension of T to the $\frac{1}{p}$ -th power space $X(\mu)^{\frac{1}{p}}$ of $X(\mu)$. This is equivalent to the existence of a constant $C > 0$ satisfying that

$$\|T(f)\| \leq C \| |f|^{\frac{1}{p}} \|_{X(\mu)}^p = C \|f\|_{X(\mu)^{\frac{1}{p}}}$$

for all $f \in X(\mu)$. The main characterization of this class of operators establishes that any of them can be extended to an space L^p of a vector measure $m_T: \Sigma \rightarrow E$ associated to T via $m_T(A) = T(\chi_A)$ and the extension is maximal. Note that the

Date: November 10, 2015.

2010 Mathematics Subject Classification. 46E30, 46G10, 47B38.

Key words and phrases. Integration with respect to a vector measure defined on a δ -ring, optimal extension of an operator, p -th power factorable operator, quasi-Banach function space.

The first author gratefully acknowledge the support of the Ministerio de Economía y Competitividad (project #MTM2012-36732-C03-03) and the Junta de Andalucía (projects FQM-262 and FQM-7276), Spain.

The second author acknowledges with thanks the support of the Ministerio de Economía y Competitividad (project #MTM2012-36740-C02-02), Spain.

condition $\chi_\Omega \in X(\mu)$ is necessary for a correct definition of p -th power factorable operator (i.e. $X(\mu) \subset X(\mu)^{\frac{1}{p}}$) and for m_T to be well defined.

Several applications are shown also in [19, § 6-7], mainly in factorization of operators through spaces $L^q(\mu)$ (Maurey-Rosenthal type theorems) and in harmonic analysis (Fourier transform and convolution operators). After that, p -th power factorable operators have turned out to be a useful tool for the study of different problems in mathematical analysis, regarding for example Banach space interpolation theory [6], differential equations [10], description of maximal domains for several classes of operators [12], factorization of kernel operators [13] or adjoint operators [11].

The requirement $\chi_\Omega \in X(\mu)$ excludes basic spaces as $L^q(0, \infty)$ or ℓ^q . Although these spaces can be represented as spaces satisfying the needed requirement (for instance $L^q(0, \infty)$ is isometrically isomorphic to $L^q(e^{-x}dx)$ via the multiplication operator induced by $e^{\frac{x}{q}}$), to use such a representation provides some kind of factorization for T but not genuine extensions.

The aim of this paper is to extend the results on maximal extensions of p -th power factorable operators to quasi-Banach spaces $X(\mu)$ which do not necessarily contain χ_Ω . Also we will consider p to be any positive number removing the restriction $p \geq 1$. The first problem is the definition of p -th power factorable operator, as in general the containment $X(\mu) \subset X(\mu)^{\frac{1}{p}}$ does not hold. This can be solved by replacing $X(\mu)^{\frac{1}{p}}$ by the sum $X(\mu)^{\frac{1}{p}} + X(\mu)$. The second problem is the definition of the vector measure m_T associated to T . The technique to overcome this obstacle consists of considering m_T defined on the δ -ring $\Sigma_{X(\mu)} = \{A \in \Sigma : \chi_A \in X(\mu)\}$ instead of the σ -algebra Σ . We will see that actually no topology is needed on $X(\mu)$ to extend $T: X(\mu) \rightarrow E$, it suffices an ideal structure on $X(\mu)$ and a certain property on T which relates the μ -a.e. pointwise order of $X(\mu)$ and the weak topology of E . This property, called order-w continuity, is the minimal condition for m_T to be a vector measure.

The paper is organized as follows. Section 2 is devoted to establish the notation and to state the results on ideal function spaces, quasi-Banach function spaces and integration with respect to a vector measure defined on a δ -ring, which will be used along this work. For the aim of completeness, we include the proof of some relevant facts. In Section 3 we show that every order-w continuous operator T defined on an ideal function space $X(\mu)$, can be extended to the space $L^1(m_T)$ of integrable functions with respect to m_T and this space is the largest one to which T can be extended as an order-w continuous operator (Theorem 3.2). Section 4 deals with operators T which are p -th power factorable with an order-w continuous extension, that is, there is an order-w continuous extension of T to the space $X(\mu)^{\frac{1}{p}} + X(\mu)$. We prove that the space $L^p(m_T) \cap L^1(m_T)$ is the optimal domain for T preserving the property of being p -th power factorable with an order-w continuous extension

(Theorem 4.2). In Sections 5 and 6 we endow $X(\mu)$ with a topology (namely, $X(\mu)$ will be a σ -order continuous quasi-Banach function space) and consider T to be continuous. Results on maximal extensions analogous to the ones of the previous sections are obtain for continuity instead of order-w continuity (Theorems 5.1 and 6.2). Finally, as an application of our results, in the last section we study when an infinite matrix of real numbers defines a continuous linear operator from ℓ^p into any given sequence space.

2. PRELIMINARIES

2.1. Ideal function spaces. Let (Ω, Σ) be a fixed measurable space. For a measure $\mu: \Sigma \rightarrow [0, \infty]$, we denote by $L^0(\mu)$ the space of all $(\mu$ -a.e. classes of) Σ -measurable real valued functions on Ω . Given two set functions $\mu, \lambda: \Sigma \rightarrow [0, \infty]$ we will write $\lambda \ll \mu$ if $\mu(A) = 0$ implies $\lambda(A) = 0$. We will say that μ and λ are *equivalent* if $\lambda \ll \mu$ and $\mu \ll \lambda$. In the case when μ and λ are two measures with $\lambda \ll \mu$, the map $[i]: L^0(\mu) \rightarrow L^0(\lambda)$ which takes a μ -a.e. class in $L^0(\mu)$ represented by f into the λ -a.e. class represented by the same f , is a well defined linear map. In order to simplify notation $[i](f)$ will be denoted again as f . Note that if λ and μ are equivalent then $L^0(\mu) = L^0(\lambda)$ and $[i]$ is the identity map i .

An *ideal function space* (briefly, i.f.s.) is a vector space $X(\mu) \subset L^0(\mu)$ satisfying that if $f \in X(\mu)$ and $g \in L^0(\mu)$ with $|g| \leq |f|$ μ -a.e. then $g \in X(\mu)$. We will say that $X(\mu)$ has the σ -property if there exists $(\Omega_n) \subset \Sigma$ such that $\Omega = \cup \Omega_n$ and $\chi_{\Omega_n} \in X(\mu)$ for all n . For instance, this happens if there is some $g \in X(\mu)$ with $g > 0$ μ -a.e.

Lemma 2.1. *Let $X(\mu)$ be an i.f.s. satisfying the σ -property. For every Σ -measurable function $f: \Omega \rightarrow [0, \infty)$ there exists $(f_n) \subset X(\mu)$ such that $0 \leq f_n \uparrow f$ pointwise.*

Proof. Let $(\Omega_n) \subset \Sigma$ be the sequence given by the σ -property of $X(\mu)$ and let $f: \Omega \rightarrow [0, \infty)$ be a Σ -measurable function. Taking $A_n = \cup_{j=1}^n \Omega_j \cap \{\omega \in \Omega : f(\omega) \leq n\}$, we have that $f_n = f\chi_{A_n} \in X(\mu)$, as $0 \leq f_n \leq n\chi_{\cup_{j=1}^n \Omega_j}$ pointwise, and that $f_n \uparrow f$ pointwise. \square

The *sum* of two i.f.s.' $X(\mu)$ and $Y(\mu)$ is the space defined as

$$X(\mu) + Y(\mu) = \{f \in L^0(\mu) : f = f_1 + f_2 \text{ } \mu\text{-a.e., } f_1 \in X(\mu), f_2 \in Y(\mu)\}.$$

Proposition 2.2. *The sum $X(\mu) + Y(\mu)$ of two i.f.s.' is an i.f.s.*

Proof. Let $f \in X(\mu) + Y(\mu)$ and $g \in L^0(\mu)$ be such that $|g| \leq |f|$ μ -a.e. Write $f = f_1 + f_2$ μ -a.e. with $f_1 \in X(\mu)$ and $f_2 \in Y(\mu)$ and denote $A = \{\omega \in \Omega : |g(\omega)| \leq |f_1(\omega)|\}$. Taking $h_1 = |g|\chi_A + |f_1|\chi_{\Omega \setminus A}$ and $h_2 = (|g| - |f_1|)\chi_{\Omega \setminus A}$, we have that $|g| = h_1 + h_2$ with $h_1 \in X(\mu)$ as $0 \leq h_1 \leq |f_1|$ pointwise and $h_2 \in Y(\mu)$ as $0 \leq h_2 \leq |f_2|$ μ -a.e. Now, denote $B = \{\omega \in \Omega : g(\omega) \geq 0\}$ and take $g_1 = h_1(\chi_B - \chi_{\Omega \setminus B})$ and

$g_2 = h_2(\chi_B - \chi_{\Omega \setminus B})$. Then, $g = g_1 + g_2$ with $g_1 \in X(\mu)$ as $|g_1| = h_1$ and $g_2 \in Y(\mu)$ as $|g_2| = h_2$. So, $g \in X(\mu) + Y(\mu)$. \square

Let $p \in (0, \infty)$. The p -power of an i.f.s. $X(\mu)$ is the i.f.s. defined as

$$X(\mu)^p = \{f \in L^0(\mu) : |f|^p \in X(\mu)\}.$$

Lemma 2.3. *Let $X(\mu)$ be an i.f.s. For $s, t \in (0, \infty)$ and $\frac{1}{r} = \frac{1}{s} + \frac{1}{t}$, it follows that if $f \in X(\mu)^s$ and $g \in X(\mu)^t$ then $fg \in X(\mu)^r$. In particular, if $\chi_\Omega \in X(\mu)$ then $X(\mu)^q \subset X(\mu)^p$ for all $0 < p < q < \infty$.*

Proof. For the first part only note that for every $a, b > 0$ it follows

$$a^r b^r \leq \frac{r}{s} a^s + \frac{r}{t} b^t. \quad (2.1)$$

For the second part take $r = p$, $s = q$ and $t = \frac{pq}{q-p}$. Then, if $f \in X(\mu)^q$, since $\chi_\Omega \in X(\mu)^t$, we have that $f = f\chi_\Omega \in X(\mu)^p$. \square

Recall that a *quasi-norm* on a real vector space X is a non-negative real map $\|\cdot\|_X$ on X satisfying

- (i) $\|x\|_X = 0$ if and only if $x = 0$,
- (ii) $\|\alpha x\|_X = |\alpha| \cdot \|x\|_X$ for all $\alpha \in \mathbb{R}$ and $x \in X$, and
- (iii) there exists a constant $K \geq 1$ such that $\|x + y\|_X \leq K(\|x\|_X + \|y\|_X)$ for all $x, y \in X$.

A quasi-norm $\|\cdot\|_X$ induces a metric topology on X in which a sequence (x_n) converges to x if and only if $\|x - x_n\|_X \rightarrow 0$. If X is complete under this topology then it is called a *quasi-Banach space* (*Banach space* if $K = 1$). A linear map $T: X \rightarrow Y$ between quasi-Banach spaces is continuous if and only if there exists a constant $M > 0$ such that $\|T(x)\|_Y \leq M\|x\|_X$ for all $x \in X$. For issues related to quasi-Banach spaces see [14].

A *quasi-Banach function space* (quasi-B.f.s. for short) is a i.f.s. $X(\mu)$ which is also a quasi-Banach space with a quasi-norm $\|\cdot\|_{X(\mu)}$ compatible with the μ -a.e. pointwise order, that is, if $f, g \in X(\mu)$ are such that $|f| \leq |g|$ μ -a.e. then $\|f\|_{X(\mu)} \leq \|g\|_{X(\mu)}$. When the quasi-norm is a norm, $X(\mu)$ is called a *Banach function space* (B.f.s.). Note that every quasi-B.f.s. is a quasi-Banach lattice for the μ -a.e. pointwise order satisfying that if $f_n \rightarrow f$ in quasi-norm then there exists a subsequence $f_{n_j} \rightarrow f$ μ -a.e. Also note that every positive linear operator between quasi-Banach lattices is continuous, see the argument given in [16, p. 2] for Banach lattices which can be adapted for quasi-Banach spaces. Then all “inclusions” of the type $[i]$ between quasi-B.f.s.’ are continuous.

A quasi-B.f.s. $X(\mu)$ is said to be σ -order continuous if for every $(f_n) \subset X(\mu)$ with $f_n \downarrow 0$ μ -a.e. it follows that $\|f_n\|_X \downarrow 0$.

It is routine to check that the intersection $X(\mu) \cap Y(\mu)$ of two quasi-B.f.s.' (B.f.s.) $X(\mu)$ and $Y(\mu)$ is a quasi-B.f.s. (B.f.s.) endowed with the quasi-norm (norm)

$$\|f\|_{X(\mu) \cap Y(\mu)} = \max \{ \|f\|_{X(\mu)}, \|f\|_{Y(\mu)} \}.$$

Moreover, if $X(\mu)$ and $Y(\mu)$ are σ -order continuous then $X(\mu) \cap Y(\mu)$ is σ -order continuous.

Proposition 2.4. *The sum $X(\mu) + Y(\mu)$ of two quasi-B.f.s.' (B.f.s.) $X(\mu)$ and $Y(\mu)$ is a quasi-B.f.s. (B.f.s.) endowed with the quasi-norm (norm)*

$$\|f\|_{X(\mu)+Y(\mu)} = \inf (\|f_1\|_{X(\mu)} + \|f_2\|_{Y(\mu)}),$$

where the infimum is taken over all possible representations $f = f_1 + f_2$ μ -a.e. with $f_1 \in X(\mu)$ and $f_2 \in Y(\mu)$. Moreover, if $X(\mu)$ and $Y(\mu)$ are σ -order continuous then $X(\mu) + Y(\mu)$ is also σ -order continuous.

Proof. From Proposition 2.2 we have that $X(\mu) + Y(\mu)$ is a i.f.s. Even more, looking at the proof we see that for every $f \in X(\mu) + Y(\mu)$ and $g \in L^0(\mu)$ with $|g| \leq |f|$ μ -a.e., if $f = f_1 + f_2$ μ -a.e. with $f_1 \in X(\mu)$ and $f_2 \in Y(\mu)$ then there exist $g_1 \in X(\mu)$ and $g_2 \in Y(\mu)$ such that $|g_i| \leq |f_i|$ μ -a.e. and $g = g_1 + g_2$. Then,

$$\|g\|_{X(\mu)+Y(\mu)} \leq \|g_1\|_{X(\mu)} + \|g_2\|_{Y(\mu)} \leq \|f_1\|_{X(\mu)} + \|f_2\|_{Y(\mu)}$$

and so, taking infimum over all possible representations $f = f_1 + f_2$ μ -a.e. with $f_1 \in X(\mu)$ and $f_2 \in Y(\mu)$, it follows that $\|g\|_{X(\mu)+Y(\mu)} \leq \|f\|_{X(\mu)+Y(\mu)}$. Hence, $\|\cdot\|_{X(\mu)+Y(\mu)}$ is compatible with the μ -a.e. pointwise order.

The proof of the fact that $\|\cdot\|_{X(\mu)+Y(\mu)}$ is a quasi-norm for which $X(\mu) + Y(\mu)$ is complete is similar to the one given in [1, § 3, Theorem 1.3] for compatible couples of Banach spaces.

Suppose that $X(\mu)$ and $Y(\mu)$ are σ -order continuous. Let $(f_n) \subset X(\mu) + Y(\mu)$ be such that $f_n \downarrow 0$ μ -a.e. Consider $f_1 = g + h$ μ -a.e. with $g \in X(\mu)$ and $h \in Y(\mu)$. We can rewrite $f_1 = f_1^1 + f_1^2$ with $f_1^1 \in X(\mu)$, $f_1^2 \in Y(\mu)$ and $f_1^1, f_1^2 \geq 0$ μ -a.e. This can be done by taking $A = \{\omega \in \Omega : f_1(\omega) \leq |g(\omega)|\}$, $f_1^1 = f_1 \chi_A + |g| \chi_{\Omega \setminus A}$ and $f_1^2 = (f_1 - |g|) \chi_{\Omega \setminus A}$. Note that $f_1^1 \in X(\mu)$ as $0 \leq f_1^1 \leq |g|$ μ -a.e. and $f_1^2 \in Y(\mu)$ as $0 \leq f_1^2 \leq |h|$ μ -a.e. Since $0 \leq f_2 \leq f_1$ μ -a.e., looking again at the proof of Proposition 2.2 we see that there exist $f_2^1 \in X(\mu)$ and $f_2^2 \in Y(\mu)$ such that $0 \leq f_2^i \leq f_1^i$ μ -a.e. and $f_2 = f_2^1 + f_2^2$ μ -a.e. By induction we construct two μ -a.e. pointwise decreasing sequences of positive functions $(f_n^1) \subset X(\mu)$ and $(f_n^2) \subset Y(\mu)$ such that $f_n = f_n^1 + f_n^2$. Note that $f_n^i \downarrow 0$ μ -a.e. as $0 \leq f_n^i \leq f_n$ μ -a.e. Then, since $X(\mu)$ and $Y(\mu)$ are σ -order continuous, we have that

$$\|f_n\|_{X(\mu)+Y(\mu)} \leq \|f_n^1\|_{X(\mu)} + \|f_n^2\|_{Y(\mu)} \rightarrow 0.$$

□

Let $p \in (0, \infty)$. The p -power $X(\mu)^p$ of a quasi-B.f.s. $X(\mu)$ is a quasi-B.f.s. endowed with the quasi-norm

$$\|f\|_{X(\mu)^p} = \| |f|^p \|_{X(\mu)}^{\frac{1}{p}}.$$

Moreover, $X(\mu)^p$ is σ -order continuous whenever $X(\mu)$ is so. Note that in the case when $X(\mu)$ is a B.f.s. and $p \geq 1$ it follows that $\|\cdot\|_{X(\mu)^p}$ is a norm and so $X(\mu)^p$ is a B.f.s. An exhaustive study of the space $X(\mu)^p$ can be found in [19, § 2.2] for the case when μ is finite and $\chi_\Omega \in X(\mu)$. This study can be extended to our general case adapting the arguments with the natural modifications (note that our p -powers here are the $\frac{1}{p}$ -th powers there).

2.2. Integration with respect to a vector measure defined on a δ -ring.

Let \mathcal{R} be a δ -ring of subsets of a set Ω , that is, a ring closed under countable intersections. Measurability will be considered with respect to the σ -algebra \mathcal{R}^{loc} of all subsets A of Ω such that $A \cap B \in \mathcal{R}$ for all $B \in \mathcal{R}$. Let us write $\mathcal{S}(\mathcal{R})$ for the space of all \mathcal{R} -simple functions, that is, simple functions with support in \mathcal{R} .

A set function $m: \mathcal{R} \rightarrow E$ with values in a Banach space E is said to be a *vector measure* if $\sum m(A_n)$ converges to $m(\cup A_n)$ in E for every sequence of pairwise disjoint sets $(A_n) \subset \mathcal{R}$ with $\cup A_n \in \mathcal{R}$.

Consider first a *real measure* $\lambda: \mathcal{R} \rightarrow \mathbb{R}$. The *variation* of λ is the measure $|\lambda|: \mathcal{R}^{loc} \rightarrow [0, \infty]$ defined as

$$|\lambda|(A) = \sup \left\{ \sum |\lambda(A_j)| : (A_j) \text{ finite disjoint sequence in } \mathcal{R} \cap 2^A \right\}.$$

Note that $|\lambda|$ is finite on \mathcal{R} . The space $L^1(\lambda)$ of integrable functions with respect to λ is defined as the classical space $L^1(|\lambda|)$. The integral with respect to λ of $\varphi = \sum_{j=1}^n \alpha_j \chi_{A_j} \in \mathcal{S}(\mathcal{R})$ over $A \in \mathcal{R}^{loc}$ is defined in the natural way by $\int_A \varphi d\lambda = \sum_{j=1}^n \alpha_j \lambda(A_j \cap A)$. The space $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\lambda)$, allowing to define the integral of $f \in L^1(\lambda)$ over $A \in \mathcal{R}^{loc}$ as $\int_A f d\lambda = \lim \int_A \varphi_n d\lambda$ for any sequence $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ converging to f in $L^1(\lambda)$.

Let now $m: \mathcal{R} \rightarrow E$ be a vector measure. The *semivariation* of m is the set function $\|m\|: \mathcal{R}^{loc} \rightarrow [0, \infty]$ defined by

$$\|m\|(A) = \sup_{x^* \in B_{E^*}} |x^* m|(A).$$

Here, B_{E^*} is the closed unit ball of the dual space E^* of E and $|x^* m|$ is the variation of the real measure $x^* m$ given by the composition of m with x^* . A set $A \in \mathcal{R}^{loc}$ is *m-null* if $\|m\|(A) = 0$, or equivalently, if $m(B) = 0$ for all $B \in \mathcal{R} \cap 2^A$. From [2, Theorem 3.2], there always exists a measure $\eta: \mathcal{R}^{loc} \rightarrow [0, \infty]$ equivalent to $\|m\|$, that is m and η have the same null sets. Let us denote $L^0(m) = L^0(\eta)$.

The space $L^1(m)$ of integrable functions with respect to m is defined as the space of functions $f \in L^0(m)$ satisfying that

- (i) $f \in L^1(x^* m)$ for every $x^* \in E^*$, and

(ii) for each $A \in \mathcal{R}^{loc}$ there exists $x_A \in E$ such that

$$x^*(x_A) = \int_A f dx^* m, \quad \text{for every } x^* \in E^*.$$

The vector x_A is unique and will be denoted by $\int_A f dm$. The space $L^1(m)$ is a σ -order continuous B.f.s. related to the measure space $(\Omega, \mathcal{R}^{loc}, \eta)$, with norm

$$\|f\|_{L^1(m)} = \sup_{x^* \in \tilde{B}_{E^*}} \int_{\Omega} |f| d|x^* m|.$$

Moreover $\mathcal{S}(\mathcal{R})$ is dense in $L^1(m)$. Note that $\int_A \varphi dm = \sum_{j=1}^n \alpha_j m(A_j \cap A)$ for every $\varphi = \sum_{j=1}^n \alpha_j \chi_{A_j} \in \mathcal{S}(\mathcal{R})$ and $A \in \mathcal{R}^{loc}$.

The integration operator $I_m: L^1(m) \rightarrow E$ defined by $I_m(f) = \int_{\Omega} f dm$ is a continuous linear operator with $\|I_m(f)\|_E \leq \|f\|_{L^1(m)}$. Even more,

$$\frac{1}{2} \|f\|_{L^1(m)} \leq \sup_{A \in \mathcal{R}} \|I_m(f \chi_A)\|_E \leq \|f\|_{L^1(m)} \quad (2.2)$$

for all $f \in L^1(m)$.

Let $p \in (0, \infty)$. We denote by $L^p(m)$ the p -power of $L^1(m)$, that is,

$$L^p(m) = \{f \in L^0(m) : |f|^p \in L^1(m)\}.$$

Then $L^p(m)$ is a quasi-B.f.s. with the quasi-norm $\|f\|_{L^p(m)} = \| |f|^p \|_{L^1(m)}^{1/p}$. In the case when $p \geq 1$, we have that $\|\cdot\|_{L^p(m)}$ is a norm and so $L^p(m)$ is a B.f.s.

These and other issues concerning integration with respect to a vector measure defined on a δ -ring can be found in [15], [17], [18], [7], [5] and [3].

3. OPTIMAL DOMAIN FOR ORDER-W CONTINUOUS OPERATORS ON A I.F.S.

Let $X(\mu)$ be a i.f.s. satisfying the σ -property (recall: $\Omega = \cup \Omega_n$ with $\chi_{\Omega_n} \in X(\mu)$ for all n) and consider the δ -ring

$$\Sigma_{X(\mu)} = \{A \in \Sigma : \chi_A \in X(\mu)\}.$$

The σ -property guarantees that $\Sigma_{X(\mu)}^{loc} = \Sigma$. Given a Banach space valued linear operator $T: X(\mu) \rightarrow E$, we define the finitely additive set function $m_T: \Sigma_{X(\mu)} \rightarrow E$ by $m_T(A) = T(\chi_A)$.

We will say that T is *order-w continuous* if $T(f_n) \rightarrow T(f)$ weakly in E whenever $f_n, f \in X(\mu)$ are such that $0 \leq f_n \uparrow f$ μ -a.e.

Proposition 3.1. *If T is order-w continuous, then m_T is a vector measure satisfying that $[i]: X(\mu) \rightarrow L^1(m_T)$ is well defined and $T = I_{m_T} \circ [i]$.*

Proof. Let $(A_n) \subset \Sigma_{X(\mu)}$ be a pairwise disjoint sequence with $\cup A_n \in \Sigma_{X(\mu)}$. Since T is order-w continuous, for any subsequence (A_{n_j}) we have that

$$\sum_{j=1}^N m_T(A_{n_j}) = T(\chi_{\cup_{j=1}^N A_{n_j}}) \rightarrow T(\chi_{\cup A_{n_j}}) = m_T(\cup A_{n_j})$$

weakly in E . From the Orlicz-Pettis theorem (see [9, Corollary I.4.4]) it follows that $\sum m_T(A_n)$ is unconditionally convergent in norm to $m_T(\cup A_n)$. Thus, m_T is a vector measure.

Note that $\|m_T\| \ll \mu$ and so $[i]: L^0(\mu) \rightarrow L^0(m_T)$ is well defined. Also, note that for every $\varphi \in \mathcal{S}(\Sigma_{X(\mu)})$ we have that $I_{m_T}(\varphi) = T(\varphi)$.

Let $f \in X(\mu)$ be such that $f \geq 0$ μ -a.e. and take a sequence of Σ -simple functions $0 \leq \varphi_n \uparrow f$ μ -a.e. For each n we can write $\varphi_n = \sum_{j=1}^m \alpha_j \chi_{A_j}$ with $(A_j)_{j=1}^m \subset \Sigma$ being a pairwise disjoint sequence and $\alpha_j > 0$ for all j . Since $\chi_{A_j} \leq \alpha_j^{-1} \varphi_n \leq \alpha_j^{-1} f$ μ -a.e., we have that $\chi_{A_j} \in X(\mu)$ and so $\varphi_n \in \mathcal{S}(\Sigma_{X(\mu)})$. Fix $x^* \in E^*$. For every $A \in \Sigma$ it follows that $x^*T(\varphi_n \chi_A) \rightarrow x^*T(f \chi_A)$ as T is order-w continuous. Note that $x^*T(\varphi_n \chi_A) = \int_A \varphi_n dx^* m_T$ and that $0 \leq \varphi_n \uparrow f$ $x^* m_T$ -a.e. as $|x^* m_T| \ll \|m_T\| \ll \mu$. From [7, Proposition 2.3], we have that $f \in L^1(x^* m_T)$ and

$$\int_A f dx^* m_T = \lim_{n \rightarrow \infty} \int_A \varphi_n dx^* m_T = \lim_{n \rightarrow \infty} x^*T(\varphi_n \chi_A) = x^*T(f \chi_A).$$

Therefore, $f \in L^1(m_T)$ and $I_{m_T}(f) = T(f)$.

For a general $f \in X(\mu)$, the result follows by taking the positive and negative parts of f . \square

For the case when $X(\mu)$ is a B.f.s., Proposition 3.1 and the next Theorem 3.2 can be deduced from [8, Proposition 2.3] and [4, Proposition 4]. The proofs given here are more direct and are valid for general i.f.s.'.

Theorem 3.2. *Suppose that T is order-w continuous. Then, T factors as*

$$\begin{array}{ccc} X(\mu) & \xrightarrow{T} & E \\ & \searrow [i] & \nearrow I_{m_T} \\ & L^1(m_T) & \end{array} \quad (3.1)$$

with I_{m_T} being order-w continuous. Moreover, the factorization is optimal in the sense:

$$(3.2) \quad \left. \begin{array}{c} \text{If } Z(\xi) \text{ is a i.f.s. such that } \xi \ll \mu \text{ and} \\ \begin{array}{ccc} X(\mu) & \xrightarrow{T} & E \\ & \searrow [i] & \nearrow S \\ & Z(\xi) & \end{array} \end{array} \right\} \Rightarrow \begin{array}{l} [i]: Z(\xi) \rightarrow L^1(m_T) \text{ is well} \\ \text{defined and } S = I_{m_T} \circ [i]. \end{array}$$

with S being an order-w continuous linear operator

Proof. The factorization (3.1) follows from Proposition 3.1. Note that the integration operator $I_{m_T}: L^1(m_T) \rightarrow E$ is order-w continuous, as it is continuous and $L^1(m_T)$ is σ -order continuous.

Let $Z(\xi)$ satisfy (3.2). In particular, $Z(\xi)$ satisfies the σ -property, as if $\chi_A \in X(\mu)$ then $\chi_A \in Z(\xi)$. From Proposition 3.1 applied to the operator $S: Z(\xi) \rightarrow E$, we have that $[i]: Z(\xi) \rightarrow L^1(m_S)$ is well defined and $S = I_{m_S} \circ [i]$. Note that $\Sigma_{X(\mu)} \subset \Sigma_{Z(\xi)}$ and $m_S(A) = S(\chi_A) = T(\chi_A) = m_T(A)$ for all $A \in \Sigma_{X(\mu)}$, that is, m_T is the restriction of $m_S: \Sigma_{Z(\xi)} \rightarrow E$ to $\Sigma_{X(\mu)}$. Then, from [4, Lemma 3], it follows that $L^1(m_S) = L^1(m_T)$ and $I_{m_S} = I_{m_T}$. \square

We can rewrite Theorem 3.2 in terms of optimal domain.

Corollary 3.3. *Suppose that T is order- w continuous. Then $L^1(m_T)$ is the largest i.f.s. to which T can be extended as an order- w continuous operator still with values in E . Moreover, the extension of T to $L^1(m_T)$ is given by the integration operator I_{m_T} .*

4. OPTIMAL DOMAIN FOR p -TH POWER FACTORABLE OPERATORS ON A I.F.S. WITH AN ORDER- w CONTINUOUS EXTENSION

Let $X(\mu)$ be a i.f.s. satisfying the σ -property and let $T: X(\mu) \rightarrow E$ be a linear operator with values in a Banach space E .

For $p \in (0, \infty)$, we call T *p -th power factorable with an order- w continuous extension* if there is an order- w continuous linear extension of T to $X(\mu)^{\frac{1}{p}} + X(\mu)$, i.e. T factors as

$$\begin{array}{ccc} X(\mu) & \xrightarrow{T} & E \\ & \searrow i & \nearrow S \\ & X(\mu)^{\frac{1}{p}} + X(\mu) & \end{array}$$

with S being an order- w continuous linear operator.

Note that in the case when $\chi_\Omega \in X(\mu)$, from Lemma 2.3, if $1 < p$ we have that $X(\mu) \subset X(\mu)^{\frac{1}{p}}$ and so $X(\mu)^{\frac{1}{p}} + X(\mu) = X(\mu)^{\frac{1}{p}}$. Similarly, if $p \leq 1$ then $X(\mu)^{\frac{1}{p}} + X(\mu) = X(\mu)$, but hence to say that T is p -th power factorable with an order- w continuous extension is just to say that T is order- w continuous.

Proposition 4.1. *The following statements are equivalent:*

- (a) *T is p -th power factorable with an order- w continuous extension.*
 - (b) *T is order- w continuous and $[i]: X(\mu)^{\frac{1}{p}} + X(\mu) \rightarrow L^1(m_T)$ is well defined.*
 - (c) *T is order- w continuous and $[i]: X(\mu) \rightarrow L^p(m_T) \cap L^1(m_T)$ is well defined.*
- Moreover, if (a)-(c) holds, the extension of T to $X(\mu)^{\frac{1}{p}} + X(\mu)$ coincides with integration operator $I_{m_T} \circ [i]$.*

Proof. (a) \Rightarrow (b) Note that T is order- w continuous as it has an order- w continuous extension. Let $S: X(\mu)^{\frac{1}{p}} + X(\mu) \rightarrow E$ be an order- w continuous linear operator extending T . Then, from Theorem 3.2, it follows that $[i]: X(\mu)^{\frac{1}{p}} + X(\mu) \rightarrow L^1(m_T)$ is well defined and $S = I_{m_T} \circ [i]$.

(b) \Leftrightarrow (c) Since T is order-w continuous, by Proposition 3.1 we always have that $[i]: X(\mu) \rightarrow L^1(m_T)$ is well defined. Suppose that $[i]: X(\mu)^{\frac{1}{p}} + X(\mu) \rightarrow L^1(m_T)$ is well defined. If $f \in X(\mu)$, since $|f|^p \in X(\mu)^{\frac{1}{p}} \subset X(\mu)^{\frac{1}{p}} + X(\mu)$, we have that $|f|^p \in L^1(m_T)$ and so $f \in L^p(m_T)$. Then $f \in L^p(m_T) \cap L^1(m_T)$. Conversely, suppose that $[i]: X(\mu) \rightarrow L^p(m_T) \cap L^1(m_T)$ is well defined. Let $f \in X(\mu)^{\frac{1}{p}} + X(\mu)$ and write $f = f_1 + f_2$ μ -a.e. with $f_1 \in X(\mu)^{\frac{1}{p}}$ and $f_2 \in X(\mu)$. Since $|f_1|^{\frac{1}{p}} \in X(\mu)$ we have that $|f_1|^{\frac{1}{p}} \in L^p(m_T) \cap L^1(m_T) \subset L^p(m_T)$ and so $f_1 \in L^1(m_T)$. Then, $f \in L^1(m_T)$ as $f_2 \in L^1(m_T)$.

(b) \Rightarrow (a) From Proposition 3.1 and since $[i]: X(\mu)^{\frac{1}{p}} + X(\mu) \rightarrow L^1(m_T)$ is well defined, we have that the operator $I_{m_T} \circ [i]$ extends T to $X(\mu)^{\frac{1}{p}} + X(\mu)$. Moreover, the extension $I_{m_T} \circ [i]: X(\mu)^{\frac{1}{p}} + X(\mu) \rightarrow E$ is order-w continuous as the integration operator $I_{m_T}: L^1(m_T) \rightarrow E$ is so. \square

In the case when $\chi_\Omega \in X(\mu)$ and T is order-w continuous, from Proposition 3.1, we have that $\chi_\Omega \in L^1(m_T)$. So, from Lemma 2.3, if $p > 1$ then $L^p(m_T) \subset L^1(m_T)$ and hence $L^p(m_T) \cap L^1(m_T) = L^p(m_T)$. If $p \leq 1$ then $L^p(m_T) \cap L^1(m_T) = L^1(m_T)$, but hence, as commented before, T being p -th power factorable with an order-w continuous extension is just T being order-w continuous.

Theorem 4.2. *Suppose that T is p -th power factorable with an order-w continuous extension. Then, T factors as*

$$\begin{array}{ccc} X(\mu) & \xrightarrow{T} & E \\ & \searrow [i] & \nearrow I_{m_T} \\ & L^p(m_T) \cap L^1(m_T) & \end{array} \quad (4.1)$$

with I_{m_T} being p -th power factorable with an order-w continuous extension. Moreover, the factorization is optimal in the sense:

$$(4.2) \quad \left. \begin{array}{ccc} X(\mu) & \xrightarrow{T} & E \\ & \searrow [i] & \nearrow S \\ & Z(\xi) & \end{array} \right\} \Rightarrow \begin{array}{l} [i]: Z(\xi) \rightarrow L^p(m_T) \cap L^1(m_T) \\ \text{is well defined and } S = I_{m_T} \circ [i]. \end{array}$$

with S being a p -th power factorable linear operator with an order-w continuous extension

Proof. The factorization (4.1) follows from Propositions 3.1 and 4.1. Note that $L^p(m_T) \cap L^1(m_T)$ satisfies the σ -property as $X(\mu)$ does. Let us see that the operator $I_{m_T}: L^p(m_T) \cap L^1(m_T) \rightarrow E$ is p -th power factorable with an order-w continuous extension by using Proposition 4.1.(c). This operator is order-w continuous as the integration operator $I_{m_T}: L^1(m_T) \rightarrow E$ is so. On other hand, since

$\Sigma_{X(\mu)} \subset \Sigma_{L^p(m_T) \cap L^1(m_T)}$ and $m_{I_{m_T}}(A) = I_{m_T}(\chi_A) = T(\chi_A) = m_T(A)$ for all $A \in \Sigma_{X(\mu)}$ (i.e. m_T is the restriction of $m_{I_{m_T}}: \Sigma_{L^p(m_T) \cap L^1(m_T)} \rightarrow E$ to $\Sigma_{X(\mu)}$), from [4, Lemma 3], it follows that $L^1(m_{I_{m_T}}) = L^1(m_T)$. Then,

$$[i]: L^p(m_T) \cap L^1(m_T) \rightarrow L^p(m_{I_{m_T}}) \cap L^1(m_{I_{m_T}}) = L^p(m_T) \cap L^1(m_T)$$

is well defined.

Let $Z(\xi)$ satisfy (4.2). In particular, $Z(\xi)$ has the σ -property. Applying Proposition 4.1 to the operator $S: Z(\xi) \rightarrow E$, we have that $[i]: Z(\xi) \rightarrow L^p(m_S) \cap L^1(m_S)$ is well defined and $S = I_{m_S} \circ [i]$. Since $\Sigma_{X(\mu)} \subset \Sigma_{Z(\xi)}$ and $m_S(A) = m_T(A)$ for all $A \in \Sigma_{X(\mu)}$, from [4, Lemma 3], it follows that $L^1(m_S) = L^1(m_T)$ and $I_{m_S} = I_{m_T}$. \square

Rewriting Theorem 4.2 in terms of optimal domain we obtain the following conclusion.

Corollary 4.3. *Suppose that T is p -th power factorable with an order- w continuous extension. Then $L^p(m_T) \cap L^1(m_T)$ is the largest i.f.s. to which T can be extended as a p -th power factorable operator with an order- w continuous extension, still with values in E . Moreover, the extension of T to $L^p(m_T) \cap L^1(m_T)$ is given by the integration operator I_{m_T} .*

5. OPTIMAL DOMAIN FOR CONTINUOUS OPERATORS ON A QUASI-B.F.S.

Let $X(\mu)$ be a quasi-B.f.s. satisfying the σ -property and let $T: X(\mu) \rightarrow E$ be a linear operator with values in a Banach space E .

Theorem 5.1. *Suppose that $X(\mu)$ is σ -order continuous and T is continuous. Then, T factors as*

$$\begin{array}{ccc} X(\mu) & \xrightarrow{T} & E \\ & \searrow [i] & \nearrow I_{m_T} \\ & L^1(m_T) & \end{array} \quad (5.1)$$

with I_{m_T} being continuous. Moreover, the factorization is optimal in the sense:

$$\left. \begin{array}{l} \text{If } Z(\xi) \text{ is a } \sigma\text{-order continuous quasi-B.f.s.} \\ \text{such that } \xi \ll \mu \text{ and} \\ \begin{array}{ccc} X(\mu) & \xrightarrow{T} & E \\ & \searrow [i] & \nearrow S \\ & Z(\xi) & \end{array} \end{array} \right\} \Rightarrow \begin{array}{l} [i]: Z(\xi) \rightarrow L^1(m_T) \text{ is well} \\ \text{defined and } S = I_{m_T} \circ [i]. \end{array}$$

(5.2)

with S being a continuous linear operator

Proof. Since $X(\mu)$ is σ -order continuous and T is continuous we have that T is order-w continuous and so the factorization (5.1) follows from Theorem 3.2. Recall that $L^1(m_T)$ is σ -order continuous and I_{m_T} is continuous.

Let $Z(\xi)$ satisfy (5.2). In particular S is order-w continuous. From Theorem 3.2 we have that $[i]: Z(\xi) \rightarrow L^1(m_T)$ is well defined and $S = I_{m_T} \circ [i]$. \square

Corollary 5.2. *Suppose that $X(\mu)$ is σ -order continuous and T is continuous. Then $L^1(m_T)$ is the largest σ -order continuous quasi-B.f.s. to which T can be extended as a continuous operator still with values in E . Moreover, the extension of T to $L^1(m_T)$ is given by the integration operator I_{m_T} .*

6. OPTIMAL DOMAIN FOR p -TH POWER FACTORABLE OPERATORS ON A QUASI-B.F.S. WITH A CONTINUOUS EXTENSION

Let $X(\mu)$ be a quasi-B.f.s. satisfying the σ -property and let $T: X(\mu) \rightarrow E$ be a linear operator with values in a Banach space E .

For $p \in (0, \infty)$, we call T p -th power factorable with a continuous extension if there is a continuous linear extension of T to $X(\mu)^{\frac{1}{p}} + X(\mu)$, i.e. T factors as

$$\begin{array}{ccc} X(\mu) & \xrightarrow{T} & E \\ & \searrow i & \nearrow S \\ & X(\mu)^{\frac{1}{p}} + X(\mu) & \end{array}$$

with S being a continuous linear operator.

Note that in the case when $\chi_\Omega \in X(\mu)$ and $1 < p$, from Lemma 2.3, it follows that $X(\mu)^{\frac{1}{p}} + X(\mu) = X(\mu)^{\frac{1}{p}}$. Then our definition of p -th power factorable operator with a continuous extension coincides with the one given in [19, Definition 5.1]. If $p \leq 1$, since $X(\mu)^{\frac{1}{p}} + X(\mu) = X(\mu)$, to say that T is p -th power factorable with a continuous extension is just to say that T is continuous.

Proposition 6.1. *Suppose that $X(\mu)$ is σ -order continuous. Then, the following statements are equivalent:*

- (a) T is p -th power factorable with a continuous extension.
- (b) T is p -th power factorable with an order-w continuous extension.
- (c) T is order-w continuous and $[i]: X(\mu)^{\frac{1}{p}} + X(\mu) \rightarrow L^1(m_T)$ is well defined.
- (d) T is order-w continuous and $[i]: X(\mu) \rightarrow L^p(m_T) \cap L^1(m_T)$ is well defined.
- (e) There exists $C > 0$ such that $\|T(f)\|_E \leq C \|f\|_{X(\mu)^{\frac{1}{p}} + X(\mu)}$ for all $f \in X(\mu)$.

Moreover, if (a)-(e) holds, the extension of T to $X(\mu)^{\frac{1}{p}} + X(\mu)$ coincides with the integration operator $I_{m_T} \circ [i]$.

Proof. (a) \Rightarrow (b) Let $S: X(\mu)^{\frac{1}{p}} + X(\mu) \rightarrow E$ be a continuous linear operator extending T . From Proposition 2.4 we have that $X(\mu)^{\frac{1}{p}} + X(\mu)$ is σ -order continuous

and so S is order-w continuous. Then, T is p -th power factorable with an order-w continuous extension.

(b) \Leftrightarrow (c) \Leftrightarrow (d) and the fact that the extension of T to $X(\mu)^{\frac{1}{p}} + X(\mu)$ coincides with the integration operator $I_{m_T} \circ [i]$ follow from Proposition 4.1.

(c) \Rightarrow (e) The operator $[i]: X(\mu)^{\frac{1}{p}} + X(\mu) \rightarrow L^1(m_T)$ is continuous as it is positive. Then, there exists a constant $C > 0$ satisfying that

$$\|f\|_{L^1(m_T)} \leq C \|f\|_{X(\mu)^{\frac{1}{p}} + X(\mu)}$$

for all $f \in X(\mu)^{\frac{1}{p}} + X(\mu)$. Since I_{m_T} extends T to $L^1(m_T)$, it follows that

$$\|T(f)\|_E = \|I_{m_T}(f)\|_E \leq \|f\|_{L^1(m_T)} \leq C \|f\|_{X(\mu)^{\frac{1}{p}} + X(\mu)}$$

for all $f \in X(\mu)$.

(e) \Rightarrow (a) Let $0 \leq f \in X(\mu)^{\frac{1}{p}} + X(\mu)$. From Lemma 2.1, there exists $(f_n) \subset X(\mu)$ such that $0 \leq f_n \uparrow f$ μ -a.e. Since $X(\mu)^{\frac{1}{p}} + X(\mu)$ is σ -order continuous, it follows that $f_n \rightarrow f$ in the quasi-norm of $X(\mu)^{\frac{1}{p}} + X(\mu)$. Then, since

$$\|T(f_n) - T(f_m)\|_E = \|T(f_n - f_m)\|_E \leq C \|f_n - f_m\|_{X(\mu)^{\frac{1}{p}} + X(\mu)},$$

we have that $(T(f_n))$ converges to some element $e \in E$. Define $S(f) = e$. Note that if $(g_n) \subset X(\mu)$ is another sequence such that $0 \leq g_n \uparrow f$ μ -a.e., then

$$\begin{aligned} \|T(f_n) - T(g_n)\|_E &\leq C \|f_n - g_n\|_{X(\mu)^{\frac{1}{p}} + X(\mu)} \\ &\leq CK \left(\|f_n - f\|_{X(\mu)^{\frac{1}{p}} + X(\mu)} + \|f - g_n\|_{X(\mu)^{\frac{1}{p}} + X(\mu)} \right), \end{aligned}$$

where K is the constant satisfying the property (iii) of the quasi-norm $\|\cdot\|_{X(\mu)^{\frac{1}{p}} + X(\mu)}$, and so S is well defined. Also note that

$$\begin{aligned} \|S(f)\|_E &\leq \|S(f) - T(f_n)\|_E + \|T(f_n)\|_E \\ &\leq \|S(f) - T(f_n)\|_E + C \|f_n\|_{X(\mu)^{\frac{1}{p}} + X(\mu)} \\ &\leq \|S(f) - T(f_n)\|_E + C \|f\|_{X(\mu)^{\frac{1}{p}} + X(\mu)} \end{aligned}$$

for all $n \geq 1$, and thus $\|S(f)\|_E \leq C \|f\|_{X(\mu)^{\frac{1}{p}} + X(\mu)}$.

For a general $f \in X(\mu)^{\frac{1}{p}} + X(\mu)$, define $S(f) = S(f^+) - S(f^-)$ where f^+ and f^- are the positive and negative parts of f respectively. It follows that S is linear and $S(f) = T(f)$ for all $f \in X(\mu)$. Moreover, for every $f \in X(\mu)^{\frac{1}{p}} + X(\mu)$ we have that

$$\begin{aligned} \|S(f)\|_E &\leq \|S(f^+)\|_E + \|S(f^-)\|_E \\ &\leq C \|f^+\|_{X(\mu)^{\frac{1}{p}} + X(\mu)} + C \|f^-\|_{X(\mu)^{\frac{1}{p}} + X(\mu)} \\ &\leq 2C \|f\|_{X(\mu)^{\frac{1}{p}} + X(\mu)}. \end{aligned}$$

an so S is continuous. Hence, T is p -th power factorable with a continuous extension. \square

In the case when μ is finite, $\chi_\Omega \in X(\mu)$ and $p \geq 1$, the equivalences (a) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e) of Proposition 6.1 are proved in [19, Theorem 5.7]. Here we have included a more detailed proof for the general case.

Theorem 6.2. *Suppose that $X(\mu)$ is σ -order continuous and T is p -th power factorable with a continuous extension. Then, T factors as*

$$\begin{array}{ccc} X(\mu) & \xrightarrow{T} & E \\ & \searrow [i] & \nearrow I_{m_T} \\ & L^p(m_T) \cap L^1(m_T) & \end{array} \quad (6.1)$$

with I_{m_T} being p -th power factorable with a continuous extension. Moreover, the factorization is optimal in the sense:

$$(6.2) \quad \left. \begin{array}{c} \text{If } Z(\xi) \text{ is a } \sigma\text{-order continuous quasi-} \\ \text{B.f.s. such that } \xi \ll \mu \text{ and} \\ \begin{array}{ccc} X(\mu) & \xrightarrow{T} & E \\ & \searrow [i] & \nearrow S \\ & Z(\xi) & \end{array} \end{array} \right\} \Rightarrow \begin{array}{l} [i]: Z(\xi) \rightarrow L^p(m_T) \cap L^1(m_T) \\ \text{is well defined and } S = I_{m_T} \circ [i]. \end{array}$$

with S being a p -th power factorable linear operator with a continuous extension

Proof. From Proposition 6.1 we have that T is p -th power factorable with an order-w continuous extension. Then, from Theorem 4.2, the factorization (6.1) holds and $I_{m_T}: L^p(m_T) \cap L^1(m_T) \rightarrow E$ is p -th power factorable with an order-w continuous extension. Noting that the space $L^p(m_T) \cap L^1(m_T)$ is σ -order continuous (as $L^1(m_T)$ is so) and satisfies the σ -property (as $X(\mu)$ does), from Proposition 6.1 it follows that $I_{m_T}: L^p(m_T) \cap L^1(m_T) \rightarrow E$ is p -th power factorable with a continuous extension.

Let $Z(\xi)$ satisfy (6.2), in particular it satisfies the σ -property. Again Proposition 6.1 gives that S is p -th power factorable with an order-w continuous extension. So, from Theorem 4.2, it follows that $[i]: Z(\xi) \rightarrow L^p(m_T) \cap L^1(m_T)$ is well defined and $S = I_{m_T} \circ [i]$. \square

Corollary 6.3. *Suppose that $X(\mu)$ is σ -order continuous and T is p -th power factorable with a continuous extension. Then $L^p(m_T) \cap L^1(m_T)$ is the largest σ -order continuous quasi-B.f.s. to which T can be extended as a p -th power factorable operator with a continuous extension, still with values in E . Moreover, the extension of T to $L^p(m_T) \cap L^1(m_T)$ is given by the integration operator I_{m_T} .*

In the case when μ is finite, $\chi_\Omega \in X(\mu)$ and $p \geq 1$, Corollary 6.3 is proved in [19, Theorem 5.11].

7. APPLICATION: EXTENSION FOR OPERATORS DEFINED ON ℓ^1

Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), c)$ where c is the counting measure on \mathbb{N} . Note that a property holds c -a.e. if and only if it holds pointwise and that the space $L^0(c)$ coincides with the space ℓ^0 of all real sequences. Consider the space $\ell^1 = L^1(c)$, which is σ -order continuous and has the σ -property. The δ -ring $\mathcal{P}(\mathbb{N})_{\ell^1}$ is just the set $\mathcal{P}_F(\mathbb{N})$ of all finite subsets of \mathbb{N} .

Let $T: \ell^1 \rightarrow E$ be a continuous linear operator with values in a Banach space E . Denote $e_n = \chi_{\{n\}}$ and assume that $T(e_n) \neq 0$ for all n . This assumption seems to be natural since if $T(e_n) = 0$ then the n -th coordinate is not involved in the action of T . Hence, the vector measure $m_T: \mathcal{P}_F(\mathbb{N}) \rightarrow E$ associated to T by $m_T(A) = T(\chi_A)$ is equivalent to c and so $L^1(m_T) \subset \ell^0$. We will write $\ell^1(m_T) = L^1(m_T)$.

Remark 7.1. By Theorem 5.1 we have that T can be extended as

$$\begin{array}{ccc} \ell^1 & \xrightarrow{T} & E \\ & \searrow i & \nearrow I_{m_T} \\ & \ell^1(m_T) & \end{array}$$

and $\ell^1(m_T)$ is the largest σ -order continuous quasi-B.f.s. to which T can be extended as a continuous operator.

Let $p > 1$. We have that T is $\frac{1}{p}$ -th power factorable with a continuous extension if there is an extension S as

$$\begin{array}{ccc} \ell^1 & \xrightarrow{T} & E \\ & \searrow i & \nearrow S \\ & \ell^p & \end{array}$$

with S being a continuous linear operator. Note that $p \leq 1$ is not considered as in this case $\ell^p \subset \ell^1$ and so the extension of T to the sum $\ell^p + \ell^1$ is just the same operator T . Applying Proposition 6.1 in the context of this section we obtain the following result.

Proposition 7.2. *The following statements are equivalent:*

- (a) T is $\frac{1}{p}$ -th power factorable with a continuous extension.
- (b) $\ell^p \subset \ell^1(m_T)$.
- (c) $\ell^1 \subset \ell^{\frac{1}{p}}(m_T) \cap \ell^1(m_T)$.
- (d) There exists $C > 0$ such that

$$\left\| \sum_{j \in M} x_j T(e_j) \right\|_E \leq C \left(\sum_{j \in M} x_j^p \right)^{\frac{1}{p}}$$

for all $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j \in M} \subset [0, \infty)$.

Proof. From Proposition 6.1, we only have to prove that condition (d) is equivalent to the following condition:

(d') There exists $C > 0$ such that $\|T(x)\|_E \leq C \|x\|_{\ell^p}$ for all $x \in \ell^1$.

If (d') holds, we obtain (d) by taking in (d') the element $x = \sum_{j \in M} x_j e_j \in \ell^1$ for every $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j \in M} \subset [0, \infty)$.

Suppose that (d) holds. Let $0 \leq x = (x_n) \in \ell^1$ and take $y^k = \sum_{j=1}^k x_j e_j$. Since $y^k \uparrow x$ pointwise, ℓ^1 is σ -order continuous and T is continuous, we have that

$$\|T(x)\|_E = \lim \|T(y^k)\|_E = \lim \left\| \sum_{j=1}^k x_j T(e_j) \right\|_E \leq C \lim \left(\sum_{j=1}^k x_j^p \right)^{\frac{1}{p}} = C \|x\|_{\ell^p}.$$

For a general $x \in \ell^1$, (d') follows by taking the positive and negative parts of x . \square

Remark 7.3. Note that if T is $\frac{1}{p}$ -th power factorable with a continuous extension then the integration operator I_{m_T} extends T to ℓ^p and, from Theorem 6.2, T factors optimally as

$$\begin{array}{ccc} \ell^1 & \xrightarrow{T} & E \\ & \searrow i & \nearrow I_{m_T} \\ & \ell^{\frac{1}{p}}(m_T) \cap \ell^1(m_T) & \end{array}$$

with I_{m_T} being $\frac{1}{p}$ -th power factorable with a continuous extension.

Now a natural question arises: When $\ell^{\frac{1}{p}}(m_T) \cap \ell^1(m_T)$ is equal to $\ell^{\frac{1}{p}}(m_T)$ or $\ell^1(m_T)$? For asking this question we introduce the following class of operators.

Let $0 < r < \infty$. We say that T is *r-power dominated* if there exists $C > 0$ such that

$$\left\| \sum_{j \in M} x_j^r T(e_j) \right\|_E^{\frac{1}{r}} \leq C \sup_{N \subset M} \left\| \sum_{j \in N} x_j T(e_j) \right\|_E$$

for every $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j \in M} \in [0, \infty)$. Note that in the case when E is a Banach lattice and T is positive we have that

$$\sup_{N \subset M} \left\| \sum_{j \in N} x_j T(e_j) \right\|_E = \left\| \sum_{j \in M} x_j T(e_j) \right\|_E.$$

Lemma 7.4. *The containment $\ell^1(m_T) \subset \ell^r(m_T)$ holds if and only if T is r-power dominated.*

Proof. Suppose that $\ell^1(m_T) \subset \ell^r(m_T)$. Since the containment is continuous (as it is positive), there exists $C > 0$ such that $\|x\|_{\ell^r(m_T)} \leq C \|x\|_{\ell^1(m_T)}$ for all $x \in \ell^1(m_T)$. For every $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j \in M} \in [0, \infty)$, we consider $x = \sum_{j \in M} x_j e_j \in \ell^1$. Noting that $x^r = \sum_{j \in M} x_j^r e_j \in \ell^1$, it follows that

$$\begin{aligned} \left\| \sum_{j \in M} x_j^r T(e_j) \right\|_E^{\frac{1}{r}} &= \|T(x^r)\|_E^{\frac{1}{r}} = \|I_{m_T}(x^r)\|_E^{\frac{1}{r}} \leq \|x^r\|_{\ell^1(m_T)}^{\frac{1}{r}} = \|x\|_{\ell^r(m_T)} \\ &\leq C \|x\|_{\ell^1(m_T)} \leq 2C \sup_{A \in \mathcal{P}_F(\mathbb{N})} \|I_{m_T}(x \chi_A)\|_E, \end{aligned}$$

where in the last inequality we have used (2.2). For every $A \in \mathcal{P}_F(\mathbb{N})$ we have that $x\chi_A = \sum_{j \in A \cap M} x_j e_j \in \ell^1$ and so $I_{m_T}(x\chi_A) = T(x\chi_A) = \sum_{j \in A \cap M} x_j T(e_j)$. Then,

$$\begin{aligned} \left\| \sum_{j \in M} x_j^r T(e_j) \right\|_E^{\frac{1}{r}} &\leq 2C \sup_{A \in \mathcal{P}_F(\mathbb{N})} \left\| \sum_{j \in A \cap M} x_j T(e_j) \right\|_E \\ &= 2C \sup_{N \subset M} \left\| \sum_{j \in N} x_j T(e_j) \right\|_E. \end{aligned}$$

Conversely, suppose that T is r -power dominated and let $x = (x_n) \in \ell^1(m_T)$. Taking $y^k = \sum_{j=1}^k |x_j|^r e_j \in \ell^1$, for every $k > \tilde{k}$ and $A \in \mathcal{P}_F(\mathbb{N})$, we have that $(y^k - y^{\tilde{k}})\chi_A = \sum_{j \in A \cap \{\tilde{k}+1, \dots, k\}} |x_j|^r e_j$ and so

$$\begin{aligned} \|T((y^k - y^{\tilde{k}})\chi_A)\|_E &= \left\| \sum_{j \in A \cap \{\tilde{k}+1, \dots, k\}} |x_j|^r T(e_j) \right\|_E \\ &\leq C^r \sup_{N \subset A \cap \{\tilde{k}+1, \dots, k\}} \left\| \sum_{j \in N} |x_j| T(e_j) \right\|_E^r \\ &= C^r \sup_{N \subset A \cap \{\tilde{k}+1, \dots, k\}} \left\| I_{m_T} \left(\sum_{j \in N} |x_j| e_j \right) \right\|_E^r \\ &\leq C^r \sup_{N \subset A \cap \{\tilde{k}+1, \dots, k\}} \left\| \sum_{j \in N} |x_j| e_j \right\|_{\ell^1(m_T)}^r \\ &\leq C^r \left\| (y^k)^{\frac{1}{r}} - (y^{\tilde{k}})^{\frac{1}{r}} \right\|_{\ell^1(m_T)}^r. \end{aligned}$$

For the last inequality note that $(y^k)^{\frac{1}{r}} = \sum_{j=1}^k |x_j| e_j$ and so

$$\sum_{j \in N} |x_j| e_j \leq \sum_{j=\tilde{k}+1}^k |x_j| e_j = (y^k)^{\frac{1}{r}} - (y^{\tilde{k}})^{\frac{1}{r}}$$

for every $N \subset A \cap \{\tilde{k}+1, \dots, k\}$. Then, by using (2.2), we have that

$$\begin{aligned} \|y^k - y^{\tilde{k}}\|_{\ell^1(m_T)} &\leq 2 \sup_{A \in \mathcal{P}_F(\mathbb{N})} \|I_{m_T}((y^k - y^{\tilde{k}})\chi_A)\|_E \\ &= 2 \sup_{A \in \mathcal{P}_F(\mathbb{N})} \|T((y^k - y^{\tilde{k}})\chi_A)\|_E \\ &\leq 2C^r \left\| (y^k)^{\frac{1}{r}} - (y^{\tilde{k}})^{\frac{1}{r}} \right\|_{\ell^1(m_T)}^r \rightarrow 0 \end{aligned}$$

as $k, \tilde{k} \rightarrow \infty$ since $(y^k)^{\frac{1}{r}} \uparrow |x|$ pointwise and $\ell^1(m_T)$ is σ -order continuous. Hence, $y^k \rightarrow z$ in $\ell^1(m_T)$ for some $z \in \ell^1(m_T)$. In particular, $y^k \rightarrow z$ pointwise and so $|x|^r = z \in \ell^1(m_T)$ as $y^k \uparrow |x|^r$ pointwise. Therefore $x \in \ell^r(m_T)$. \square

Lemma 7.5. *Let $p > 1$. If T is $\frac{1}{p}$ -power dominated then it is $\frac{1}{p}$ -th power factorable with a continuous extension.*

Proof. Let us use Proposition 7.2.(d). Given $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j \in M} \subset [0, \infty)$, denoting by K the continuity constant of T , we have that

$$\begin{aligned} \left\| \sum_{j \in M} x_j T(e_j) \right\|_E &= \left\| \sum_{j \in M} (x_j^p)^{\frac{1}{p}} T(e_j) \right\|_E \leq C^{\frac{1}{p}} \sup_{N \subset M} \left\| \sum_{j \in N} x_j^p T(e_j) \right\|_E^{\frac{1}{p}} \\ &= C^{\frac{1}{p}} \sup_{N \subset M} \left\| T \left(\sum_{j \in N} x_j^p e_j \right) \right\|_E^{\frac{1}{p}} \leq C^{\frac{1}{p}} K^{\frac{1}{p}} \sup_{N \subset M} \left\| \sum_{j \in N} x_j^p e_j \right\|_{\ell^1}^{\frac{1}{p}} \\ &= C^{\frac{1}{p}} K^{\frac{1}{p}} \sup_{N \subset M} \left(\sum_{j \in N} x_j^p \right)^{\frac{1}{p}} \leq C^{\frac{1}{p}} K^{\frac{1}{p}} \left(\sum_{j \in M} x_j^p \right)^{\frac{1}{p}}. \end{aligned}$$

□

As a consequence of Remark 7.3, Lemma 7.4 and Lemma 7.5, we obtain the following conclusion.

Corollary 7.6. *For $p > 1$ we have that:*

- (a) *If T is p -power dominated and $\frac{1}{p}$ -th power factorable with a continuous extension, then T factors optimally as*

$$\begin{array}{ccc} \ell^1 & \xrightarrow{T} & E \\ & \searrow i & \nearrow I_{m_T} \\ & \ell^{\frac{1}{p}}(m_T) & \end{array}$$

with I_{m_T} being $\frac{1}{p}$ -th power factorable with a continuous extension.

- (b) *If T is $\frac{1}{p}$ -power dominated, then T factors optimally as*

$$\begin{array}{ccc} \ell^1 & \xrightarrow{T} & E \\ & \searrow i & \nearrow I_{m_T} \\ & \ell^1(m_T) & \end{array}$$

with I_{m_T} being $\frac{1}{p}$ -th power factorable with a continuous extension.

Consider now the case when $E = \ell(c)$ is a B.f.s. related to c such that $\ell^1 \subset \ell(c) \subset \ell^0$. Then $\ell(c)$ is a *Köthe function space* in the sense of Lindenstrauss and Tzafriri, see [16, p. 28-30]. For instance, $\ell(c)$ could be an ℓ^q space with $1 \leq q \leq \infty$, or a Lorentz sequence space $\ell^{q,r}$ with $1 \leq r \leq q \leq \infty$ or an Orlicz sequence space ℓ_φ with φ being an Orlicz function.

Let us recall some facts about the Köthe dual of an space $\ell(c)$. Denote the *scalar product* of two sequences $x = (x_n), y = (y_n) \in \ell^0$ by

$$(x, y) = \sum x_n y_n$$

provided the sum exists. The *Köthe dual* of $\ell(c)$ is given by

$$\ell(c)' = \left\{ y \in \ell^0 : (|x|, |y|) < \infty \text{ for all } x \in \ell(c) \right\}.$$

Note that $\chi_A \in \ell(c)'$ for all $A \in \mathcal{P}_F(\mathbb{N})$. The space $\ell(c)'$ endowed with the norm

$$\|y\|_{\ell(c)'} = \sup_{x \in B_{\ell(c)}} (|x|, |y|)$$

is a B.f.s. in the sense of Lindenstrauss and Tzafriri. The map $j: \ell(c)' \rightarrow \ell(c)^*$ defined by $\langle j(y), x \rangle = (x, y)$ for all $y \in \ell(c)'$ and $x \in \ell(c)$, is a linear isometry. In particular, convergence in norm of $\ell(c)$ implies pointwise convergence, as $e_n \in \ell(c)'$ for all n . Note that $\ell(c) \subset \ell(c)''$. The equality $\ell(c) = \ell(c)''$ holds with equal norms if and only if $\ell(c)$ has the *Fatou property*, that is, if $(x^k) \subset \ell(c)$ is such that $0 \leq x^k \uparrow x$ pointwise and $\sup \|x^k\|_{\ell(c)} < \infty$ then $x \in \ell(c)$ and $\|x^k\|_{\ell(c)} \uparrow \|x\|_{\ell(c)}$.

Let $M = (a_{ij})$ be an infinite matrix of real numbers and denote by C_j the j -th column of M . Assume $C_j \neq 0$ for all j . Note that

$$Mx = \left(\sum_j a_{ij} x_j \right)_i$$

for any $x \in \ell^0$ for which it is meaningful to do so.

Proposition 7.7. *Suppose that $\ell(c)$ has the Fatou property. Then, the following statements are equivalent:*

- (a) M defines a continuous linear operator $M: \ell^1 \rightarrow \ell(c)$.
- (b) $C_j \in \ell(c)$ for all j and $\sup_j \|C_j\|_{\ell(c)} < \infty$.

Proof. (a) \Rightarrow (b) Let $K > 0$ be such that $\|Mx\|_{\ell(c)} \leq K\|x\|_{\ell^1}$ for all $x \in \ell^1$. For every j we have that $C_j = Me_j \in \ell(c)$. Moreover,

$$\sup_j \|C_j\|_{\ell(c)} = \sup_j \|Me_j\|_{\ell(c)} \leq K \sup_j \|e_j\|_{\ell^1} = K.$$

(b) \Rightarrow (c) Since $\ell(c)$ has the Fatou property then $\ell(c) = \ell(c)''$ with equal norms. Let $x \in \ell^1$. First note that for every i we have that

$$\begin{aligned} \sum_j |a_{ij} x_j| &= \sum_j (|C_j|, e_i) |x_j| \leq \sum_j \|C_j\|_{\ell(c)} \|e_i\|_{\ell(c)'} |x_j| \\ &\leq \|e_i\|_{\ell(c)'} \|x\|_{\ell^1} \sup_j \|C_j\|_{\ell(c)} \end{aligned}$$

and so $Mx \in \ell^0$. Given $y \in \ell(c)'$ it follows that

$$\begin{aligned} (|y|, |Mx|) &= \sum_i |y_i| \left| \sum_j a_{ij} x_j \right| \leq \sum_i \sum_j |a_{ij} x_j y_i| = \sum_j |x_j| \sum_i |a_{ij} y_i| \\ &= \sum_j |x_j| (|C_j|, |y|) \leq \sum_j |x_j| \|C_j\|_{\ell(c)} \|y\|_{\ell(c)'} \\ &\leq \|y\|_{\ell(c)'} \|x\|_{\ell^1} \sup_j \|C_j\|_{\ell(c)}. \end{aligned}$$

Then $Mx \in \ell(c)'' = \ell(c)$ and

$$\|Mx\|_{\ell(c)} = \sup_{y \in B_{\ell(c)'}} (|y|, |Mx|) \leq \|x\|_{\ell^1} \sup_j \|C_j\|_{\ell(c)}.$$

□

In what follows assume that $\ell(c)$ has the Fatou property, $C_j \in \ell(c)$ for all j and $\sup_j \|C_j\|_{\ell(c)} < \infty$. Then, M defines a continuous linear operator $M: \ell^1 \rightarrow \ell(c)$ and so, by Remark 7.1 we have that M can be extended as

$$\begin{array}{ccc} \ell^1 & \xrightarrow{M} & \ell(c) \\ & \searrow i & \nearrow I_{m_M} \\ & \ell^1(m_M) & \end{array}$$

and $\ell^1(m_M)$ is the largest σ -order continuous quasi-B.f.s. to which M can be extended as a continuous operator.

Remark 7.8. For every $x \in \ell^1(m_M)$ it follows that $I_{m_M}(x) = Mx$ and so M defines a continuous linear operator $M: \ell^1(m_M) \rightarrow \ell(c)$. Indeed, take $0 \leq x = (x_n) \in \ell^1(m_M)$ and $x^k = \sum_{j=1}^k x_j e_j \in \ell^1$. Since $x^k \uparrow x$ pointwise and $\ell^1(m_M)$ is σ -order continuous it follows that $x^k \rightarrow x$ in $\ell^1(m_M)$. Then, since $M = I_{m_M}$ on ℓ^1 , we have that $Mx^k = I_{m_M}(x^k) \rightarrow I_{m_M}(x)$ in $\ell(c)$ and so pointwise. Hence, the i -th coordinate $\sum_{j=1}^k a_{ij}x_j$ of Mx^k converges to the i -th coordinate of $I_{m_M}(x)$ and thus $Mx = I_{m_M}(x) \in \ell(c)$. For a general $x \in \ell^1(m_M)$, we only have to take the positive and negative parts of x .

From Proposition 7.2 applied to $M: \ell^1 \rightarrow \ell(c)$ and Remark 7.8 we obtain the following conclusion.

Proposition 7.9. *The following statements are equivalent:*

- (a) M defines a continuous linear operator $M: \ell^p \rightarrow \ell(c)$.
- (b) M is $\frac{1}{p}$ -th power factorable with a continuous extension.
- (c) $\ell^p \subset \ell^1(m_M)$.
- (d) $\ell^1 \subset \ell^{\frac{1}{p}}(m_M) \cap \ell^1(m_M)$.
- (e) There exists $C > 0$ such that

$$\left\| \sum_{j \in M} x_j C_j \right\|_{\ell(c)} \leq C \left(\sum_{j \in M} x_j^p \right)^{\frac{1}{p}}$$

for all $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j \in M} \subset [0, \infty)$.

Proof. The equivalence among statements (b), (c), (d), (e) is given by Proposition 7.2. The statement (a) implies (b) obviously. From Remark 7.8 we have that M defines a continuous linear operator $M: \ell^1(m_M) \rightarrow \ell(c)$, so (c) implies (a). □

Let us give two conditions guaranteeing that M defines a continuous linear operator $M: \ell^p \rightarrow \ell(c)$:

- (I) If p' is the conjugate exponent of p and $\sum \|C_j\|_{\ell(c)}^{p'} < \infty$, then (e) in Proposition 7.9 holds. Indeed, for every $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j \in M} \subset [0, \infty)$ we have that

$$\begin{aligned} \left\| \sum_{j \in M} x_j C_j \right\|_{\ell(c)} &\leq \sum_{j \in M} x_j \|C_j\|_{\ell(c)} \leq \left(\sum_{j \in M} x_j^p \right)^{\frac{1}{p}} \left(\sum_{j \in M} \|C_j\|_{\ell(c)}^{p'} \right)^{\frac{1}{p'}} \\ &\leq \left(\sum_{j \in M} \|C_j\|_{\ell(c)}^{p'} \right)^{\frac{1}{p'}} \left(\sum_{j \in M} x_j^p \right)^{\frac{1}{p}}. \end{aligned}$$

- (II) If M is $\frac{1}{p}$ -power dominated, that is, there exists $C > 0$ such that

$$\left\| \sum_{j \in M} x_j^{\frac{1}{p}} C_j \right\|_{\ell(c)}^p \leq C \sup_{N \subset M} \left\| \sum_{j \in N} x_j C_j \right\|_{\ell(c)}$$

for every $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j \in M} \in [0, \infty)$, then (b) in Proposition 7.9 holds by Lemma 7.5.

For instance, in the case when $\ell(c) = \ell^q$ and $a_{ij} \geq 0$ for all i, j , condition (II) is satisfied if $F_i \in \ell^1$ for all i and $\sum \|F_i\|_{\ell^1}^q < \infty$, where F_i denotes the i -th file of M . Indeed, for every $M \in \mathcal{P}_F(\mathbb{N})$ and $(x_j)_{j \in M} \in [0, \infty)$, applying Hölder's inequality twice for p and its conjugate exponent p' , we have that

$$\begin{aligned} \left\| \sum_{j \in M} x_j^{\frac{1}{p}} C_j \right\|_{\ell^q}^p &= \left(\sum_i \left(\sum_{j \in M} x_j^{\frac{1}{p}} a_{ij} \right)^q \right)^{\frac{p}{q}} = \left(\sum_i \left(\sum_{j \in M} x_j^{\frac{1}{p}} a_{ij}^{\frac{1}{p}} a_{ij}^{1-\frac{1}{p}} \right)^q \right)^{\frac{p}{q}} \\ &\leq \left(\sum_i \left(\sum_{j \in M} x_j a_{ij} \right)^{\frac{q}{p}} \left(\sum_{j \in M} a_{ij} \right)^{\frac{q}{p'}} \right)^{\frac{p}{q}} \\ &\leq \left(\sum_i \left(\sum_{j \in M} x_j a_{ij} \right)^q \right)^{\frac{1}{q}} \cdot \left(\sum_i \left(\sum_{j \in M} a_{ij} \right)^q \right)^{\frac{p}{qp'}} \\ &\leq \left\| \sum_{j \in M} x_j C_j \right\|_{\ell^q} \left(\sum_i \|F_i\|_{\ell^1}^q \right)^{\frac{p}{qp'}}. \end{aligned}$$

Note that $\sup_{N \subset M} \left\| \sum_{j \in N} x_j C_j \right\|_{\ell^q} = \left\| \sum_{j \in M} x_j C_j \right\|_{\ell^q}$ as $a_{ij} \geq 0$ for all i, j .

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